General Problems.

(1) P.G.-sem-2nd paper-VI, unit-IV problem of complex int. Dr. G. D. Singh

15 शनि

(1.) Obtain the Taylor's and Laurent's series which represents the function $\dfrac{z^2-1}{(z+2)(z+3)}$ in the regions

(i.) $|z|<2$ (ii.) $2<|z|<3$ (iii.) $|z|<3$

$$f(z) = \frac{z^2-1}{(z+2)(z+3)} = \frac{z^2-1}{z^2+5z+6}$$

$$z^2+5z+6 \,\big)\, z^2-1 \,\big(\, 1$$
$$\underline{z^2+5z+6}$$
$$-5z-7$$

$$\therefore f(z) = 1 + \frac{-5z-7}{z^2+5z+6}$$

16 रवि

$$= 1 - \frac{5z+7}{(z+2)(z+3)} \quad \text{(1.)}$$

Now, $\dfrac{5z+7}{(z+2)(z+3)} = \dfrac{A}{z+2} + \dfrac{B}{z+3} = \dfrac{A(z+3)+B(z+2)}{(z+2)(z+3)}$

$$\therefore 5z+7 = A(z+3) + B(z+2)$$

Let $z=-3$, $5(-3)+7 = A(0)+B(-3+2)$

$$-15+7 = -B \therefore B = 8$$

Scanned with Cam

$z = -2, \quad 5(-2)+7 = A(-2+3) + B(0)$

$-3 = A$

$\therefore$ From (1.)

$$f(z) = 1 - \left\{ \frac{-3}{z+2} + \frac{8}{z+3} \right\}$$

or, $f(z) = 1 + \dfrac{3}{z+2} - \dfrac{8}{z+3}$ ——— (2.)

(i.) when $|z| < 2$ or $\dfrac{|z|}{2} < 1$

and $\dfrac{|z|}{3} < 1$

$$\therefore f(z) = 1 + \frac{3}{2\left(1+\frac{z}{2}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{2}\left\{1 - \frac{z}{2} + \frac{z^2}{2^2} - \frac{z^3}{2^3} + \cdots\right\}$$

$$- \frac{8}{3}\left\{1 - \frac{z}{3} + \frac{z^2}{3^2} - \frac{z^3}{3^3} + \cdots\right\}$$

$$= 1 + \frac{3}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

$$= 1 + \sum_{n=0}^{\infty}(-1)^n\left\{\frac{3}{2^{n+1}} - \frac{8}{3^{n+1}}\right\}z^n \quad \text{Ans}$$

19 बुध

(ii.) when $2 < |z| < 3$

or $\frac{2}{|z|} < 1$ and $\frac{|z|}{3} < 1$

$$\therefore f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$$= 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left\{1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \cdots\right\} - \frac{8}{3}\left\{1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \cdots\right\}$$

20 गुरु

or $$f(z) = 1 + \frac{3}{z}\sum_{0}^{\infty}(-1)^n \frac{2^n}{z^n} - \frac{8}{3}\sum_{0}^{\infty}(-1)^n \frac{z^n}{3^n}$$

$$= 1 + \sum_{n=0}^{\infty}(-1)^n\left\{\frac{3\cdot 2^n}{z^{n+1}} - \frac{8\cdot z^n}{3^{n+1}}\right\}$$

This is Laurent's series in the annulus in $2<|z|<3$.

(iii.) when $|z| > 3$, $|z|>2$ then $\frac{3}{|z|} < 1$

Now $$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left[1 - \frac{2}{z} + \frac{2^2}{z^2} - \frac{2^3}{z^3} + \cdots\right] - \frac{8}{z}\left[1 - \frac{3}{z} + \frac{3^2}{z^2} - \cdots\right]$$

$$= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{z}\right)^n$$

$$= 1 + \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}}\left\{3\cdot 2^n - 3^n\cdot 8\right\}$$ Ans